Linear Algebra and its Applications 432 (2010) 1258–1266

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The μ-permanent of a tridiagonal matrix, orthogonal polynomials, and chain sequences[☆]

C.M. da Fonseca

Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

ARTICLE INFO

Article history:
Received 8 August 2009
Accepted 25 October 2009
Available online 4 December 2009

Submitted by R. Horn

AMS classification:
15A45
15A15
05C05
05C20

Keywords:
Permanent
Determinant
μ-Permanent
Orthogonal polynomials sequence
Chain sequences
Tridiagonal matrix
Toeplitz matrix

ABSTRACT

Let $A = (a_{ij})$ be an $n \times n$ complex matrix. For any real μ, define the polynomial

$$P_\mu(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \mu^{\ell(\sigma)},$$

where $\ell(\sigma)$ is the number of inversions of the permutation σ in the symmetric group S_n. We analyze and establish a conjecture on the location of the zeros of $P_\mu(A)$, when A is a non-diagonal positive definite matrix. We prove the conjecture for the particular case when A is a Jacobi matrix. Our proof is independent from known results, and uses a connection with orthogonal polynomials and chain sequences.

1. Preliminaries

For a given $n \times n$ complex matrix $A = (a_{ij})$, the *μ-permanent* of A, $P_\mu(A)$, is the polynomial of a real variable μ defined as

$$P_\mu(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right) \mu^{\ell(\sigma)},$$

☆ This work was supported by CMUC – Centro de Matemática da Universidade de Coimbra.
E-mail address: cmf@mat.uc.pt

0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.10.036

where

$$\ell(\sigma) = \#\{(i,j) \in \{1,\ldots,n\}^2 | i < j \text{ and } \sigma(i) > \sigma(j)\},$$

the number of inversions in the permutation σ of the symmetric group S_n of degree n, i.e., the number of interchanges of consecutive elements of $\sigma \in S_n$ necessary to arrange them in their natural order [15, p. 1].

The μ-permanent was introduced in a research note [1], in 1992, by Ravindra Bapat, and it can be seen as a natural generalization both of the determinant ($\mu = -1$) and of the permanent ($\mu = 1$), respectively. Note also that $P_0(A) = a_{11} \cdots a_{nn}$. In [2,11], Bapat and Lal renamed this polynomial as the q-permanent. Here, we keep the original notation used in [1].

There are other attractive matrix functions which are generalizations of these numbers associated to matrices such as the α-permanent or the immanant among others [4,13,14,17].

The μ-permanent has many interesting and rich general properties [1,2,11], but nevertheless the task of computing this polynomial for a given matrix is tricky and, in general, there is no easy way to compute it. Thus, the μ-permanent is in this regard a more intractable object than the determinant or the permanent.

Using the positive definiteness of the function f defined on S_n by $f(\sigma) = \mu^{\ell(\sigma)}$, proved by Bożejko and Speicher in [3], Bapat showed in [1] that for any Hermitian positive semidefinite matrix A,

$$P_\mu(A) \geqslant 0, \quad \text{if } \mu \in [-1, 1],$$

and conjectured (and proved for $n \leqslant 3$) that $P_\mu(A)$ is a strictly increasing function of $\mu \in [-1, 1]$, assuming additionally that A is non-diagonal. The veracity of the conjecture implies that

$$\det A \leqslant P_\mu(A) \leqslant \operatorname{per} A,$$

which provides a generalization of both the classical Hadamard inequality and the permanental analogue proved by Marcus [12].

The stated conjecture has been proved for an irreducible tridiagonal positive definite matrix in [10], and independently by the author in [7] as an immediate consequence of the following theorem.

Theorem 1.1 [7]. *For a given positive definite matrix A, whose graph is a tree, $P_\mu(A)$ is a strictly increasing function of $\mu \in [-1, 1]$.*

The main thrust of this note is of a different nature. We extend Bapat's conjecture to the real line and prove it for the tridiagonal case, applying an important notion commonly used in the theory of orthogonal polynomials: chain sequences. In the end, a particular case will be considered.

For the sake of simplicity, we deal with real symmetric matrices. We fix the notation $A > 0$ for a real symmetric positive definite matrix A.

2. A conjecture

Since we already have some results and conjectures for $P_\mu(A)$, for $\mu \in [-1, 1]$, we may wonder what happens when μ runs over all the real line. After some numerical computations and taking into account known results, we were led to the following conjecture.

Conjecture 1. *For a given matrix $A > 0$, there exists $\epsilon \leqslant -1$ such that $P_\mu(A)$ is a strictly increasing function of $\mu \in (\epsilon, +\infty)$.*

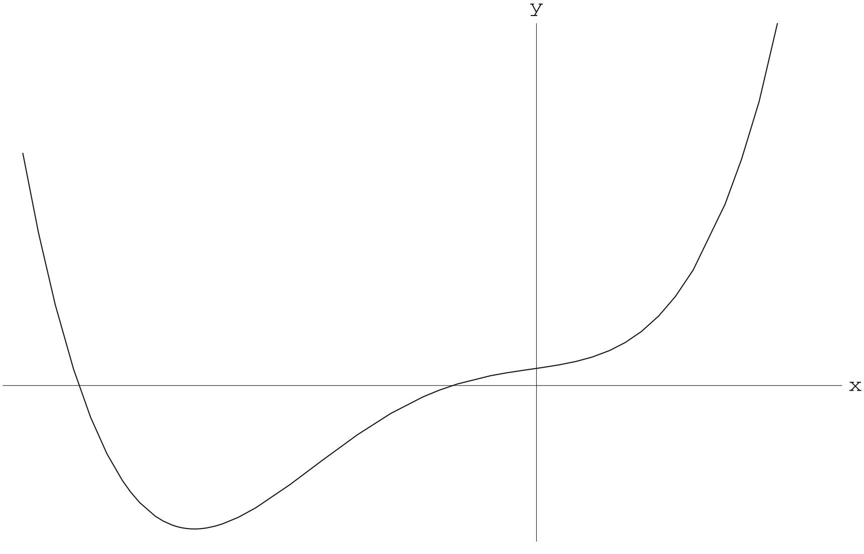
Let us consider the symmetric positive definite matrix

$$A = \begin{pmatrix} 3 & -1 & 0 & -2 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 3 & -1 \\ -2 & 1 & -1 & 2 \end{pmatrix}.$$

Then

$$P_\mu(A) = 4\mu^6 + 12\mu^5 + 12\mu^4 + 13\mu^3 + 7\mu^2 + 15\mu + 18.$$

The real zeros of $P_\mu(A)$ are -2.21893 and -1.03484 and the graph of $P_\mu(A)$ is



If this conjecture is true, then all real zeros of $P_\mu(A)$ are located in a interval (a, b), with $-\infty < a < b < -1$, since $P_{-1}(A) = \det A > 0$. Naturally, if the degree of $P_\mu(A)$ is odd, there exists always at least one real zero.

Considering the matrix

$$B = \begin{pmatrix} 3 & -1 & 0 & -2 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 3 & -1 \\ -2 & 1 & -1 & 1 \end{pmatrix},$$

we have

$$P_\mu(B) = 4\mu^6 + 12\mu^5 + 12\mu^4 + 13\mu^3 + 7\mu^2 + 9\mu + 9$$

with real zeros -2.19512 and -0.900334. Observe that B is not positive definite.

Before we turn our attention to the tridiagonal case, we recall a little-known but important concept connected with orthogonal polynomials.

3. Orthogonal polynomials and chain sequences

The theory of orthogonal polynomials has extensive applications to various branches of physics, probability and statistics, and mathematics. Recall that a sequence of real polynomials $\{P_n(x)\}_{n\geqslant 0}$ is *orthogonal* with respect to a distribution function ψ, i.e., a nondecreasing function defined on the real line $\mathbb{R}$ whose moments of all orders exist, if each $P_n(x)$ is of precise degree n, and

$$0 < \int_{\mathbb{R}} P_n^2(x)\, d\psi(x) < \infty \quad \text{and} \quad \int_{\mathbb{R}} P_n(x)P_m(x)\, d\psi(x) = 0,$$

for every $n, m \geqslant 0$, with $n \neq m$ [16, Chapter II]. A real monic orthogonal polynomial sequence will be abbreviate to MOPS.

The celebrated *Favard Theorem* [6] (also known as the *Spectral Theorem* for orthogonal polynomials), asserts a significant characterization of a MOPS $\{P_n(x)\}_{n\geqslant 0}$. In fact, $\{P_n(x)\}_{n\geqslant 0}$ is a MOPS if and only if it satisfies the classical three-term recurrence formula

$$xP_n(x) = P_{n+1}(x) + c_nP_n(x) + \gamma_nP_{n-1}(x), \quad n = 0, 1, 2, \ldots \tag{3.1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, and where c_n is real and γ_n is positive, for all positive integers n.

The smallest interval containing in its interior all zeros of all $P_n(x)$, say $[\xi, \eta]$, is called the *true interval of orthogonality*, it being known that at least one admissible $d\psi(x)$ has its support restricted to this interval. Allowing $\xi = -\infty$ and $\eta = +\infty$, the limits of the smallest and largest zeros of $\{P_n(x)\}_{n\geqslant 0}$ are ξ and η, respectively.

Perhaps the most familiar example of OPS is the Chebyshev polynomials of second kind, $\{U_n(x)\}_{n\geqslant 0}$, satisfying the three-term recurrence relations

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x),$$

for all $n = 1, 2, \ldots$, with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. Each $U_n(x)$ also satisfies

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta \ (0 \leqslant \theta < \pi)$$

for all $n = 0, 1, 2 \ldots$, from which one deduces the orthogonality relations

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\, dx = \frac{\pi}{2}\delta_{n,m}.$$

The zeros of $U_n(x)$ are $\cos\left(\frac{\ell\pi}{n+1}\right)$, for $\ell = 1, 2, \ldots, n$ [8], and the true interval of orthogonality of $\{U_n(x)\}_{n\geqslant 0}$ is $[-1, 1]$.

Definition 3.1. A sequence $\{a_n\}_{n\geqslant 0}$ is a (*positive*) *chain sequence* if there is a parameter sequence $\{g_n\}_{n\geqslant 0}$ such that

$$a_n = g_n(1 - g_{n-1}), \quad \text{with } 0 \leqslant g_0 < 1 \text{ and } 0 < g_n < 1, \ \text{for } n > 0.$$

This concept was introduced by Hubert Wall in connection with continued fractions [18, p. 79]. Chain sequences also provide a very useful tool to find bounds on the spectrum of a special stationary Markov processes whose state space is the nonnegative integers, known as birth and death processes [9].

Setting

$$\alpha_n(x) = \frac{\gamma_n}{(x - c_{n-1})(x - c_n)}, \quad \text{for } n > 0,$$

Chihara proved in [5] the following theorem (also [9, Theorem 5.3]).

Theorem 3.1. *Let $\{P_n(x)\}_{n\geqslant 0}$ be a real MOPS generated by (3.1). Then an interval $[a, b]$ contains the true interval of orthogonality of $\{P_n(x)\}_{n\geqslant 0}$ if and only if*

$$a < c_n < b$$

for $n > 0$, and if

$$\{\alpha_n(a)\}_{n\geqslant 0} \quad \text{and} \quad \{\alpha_n(b)\}_{n\geqslant 0}$$

are chain sequences.

Among other results and conjectures, Chihara showed that $P_n(x)$ are the orthogonal polynomials with the support of $d\psi(x)$ restricted to the non-negative real line if and only if these same conditions on the λ_n and c_n hold with the additional restriction that the chain sequence does not uniquely determine its parameters.

The following consequence of Theorem 3.1 plays a fundamental role in this work.

Corollary 3.2. *Let* $\{P_n(x)\}_{n \geqslant 0}$ *be a real MOPS generated by* (3.1), *with* $c_n = 0$. *Then the interval* $[-1, 1]$ *contains the true interval of orthogonality of* $\{P_n(x)\}_{n \geqslant 0}$ *if and only if*

$$\{\alpha_n(-1)\}_{n \geqslant 0} \quad \textit{and} \quad \{\alpha_n(1)\}_{n \geqslant 0}$$

are chain sequences.

4. The μ-permanent of a tridiagonal matrix

We will now confine our attention on (irreducible) real, symmetric tridiagonal matrices

$$A_n = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix}, \tag{4.1}$$

where $a_i > 0$, for $i = 1, \ldots, n$. In this section we prove Conjecture 1 when $A_n > 0$ is of the form (4.1).

In general, from [7], we have the three-term recurrence relation

$$P_\mu(A_n) = a_n P_\mu(A_{n-1}) + b_{n-1}^2 P_\mu(A_{n-2})\mu \tag{4.2}$$

with initial conditions $P_\mu(A_0) = 1$ and $P_\mu(A_1) = a_1$. We point out that the sign of the off-diagonal entries is irrelevant. The degree of $P_\mu(A_n)$ is $\ell(\tau)$, where $\tau = (1\,2)\cdots(n-1\;n)$, if n is even, and $\tau = (2\,3)\cdots(n-1\;n)$, otherwise.

Lemma 4.1. *The degree of* $P_\mu(A_n)$ *is*

$$\left\lfloor \frac{n}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ *denotes the integer part of* x.

When n is even, the leading coefficient of $P_\mu(A_n)$ is

$$b_1^2 b_3^2 \cdots b_{n-1}^2,$$

and, when $n = 2k + 1$ is odd, is

$$\sum_{i=0}^{k} b_1^2 b_3^2 \cdots b_{2i-1}^2 a_{2i+1} b_{2i+2}^2 b_{2i+4}^2 \cdots b_{n-1}^2.$$

Still for n odd, if $a_i = 0$, for $i = 1, \ldots, n$, then $P_\mu(A_n) = 0$.

Remark 4.1. The sequence of polynomials satisfying (4.2) is not an OPS.

Making

$$Q_n(\mu) = \frac{1}{\mu^n} P_{-\mu^2}(A_n), \tag{4.3}$$

from (4.2), we get

$$Q_n(x) = \frac{a_n}{x} Q_{n-1}(x) - b_{n-1}^2 Q_{n-2}(x), \tag{4.4}$$

with initial conditions $Q_0(x) = 1$ and $Q_1(x) = a_1$.

Assuming that all diagonal entries are positive, (4.4) is equivalent to

$$\frac{1}{x}Q_{n-1}(x) = \frac{1}{a_n}Q_n(x) + \frac{b_{n-1}^2}{a_n}Q_{n-2}(x). \tag{4.5}$$

Remark 4.2. Each $Q_n(x)$ is symmetric, i.e., $(-1)^n Q_n(x) = Q_n(-x)$.

We can convert (4.5) into the matricial recurrence relation

$$\begin{pmatrix} 0 & \frac{1}{a_1} & & & \\ \frac{b_1^2}{a_2} & 0 & \frac{1}{a_2} & & \\ & \frac{b_2^2}{a_3} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{a_{n-1}} \\ & & & \frac{b_{n-1}^2}{a_n} & 0 \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \\ Q_{n-1}(x) \end{pmatrix} + \frac{1}{a_n}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ Q_n(x) \end{pmatrix} = \frac{1}{x}\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \\ Q_{n-1}(x) \end{pmatrix}. \tag{4.6}$$

Therefore, the zeros of $Q_n(x)$ are the reciprocals of the (nonzero) eigenvalues of the main matrix in (4.6), or, equivalently, of the Jacobi matrix

$$J_n = \begin{pmatrix} 0 & \frac{b_1}{\sqrt{a_1a_2}} & & & \\ \frac{b_1}{\sqrt{a_1a_2}} & 0 & \frac{b_2}{\sqrt{a_2a_3}} & & \\ & \frac{b_2}{\sqrt{a_2a_3}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{b_{n-1}}{\sqrt{a_{n-1}a_n}} \\ & & & \frac{b_{n-1}}{\sqrt{a_{n-1}a_n}} & 0 \end{pmatrix}, \tag{4.7}$$

and are all real.

The analysis of whether a symmetric tridiagonal matrix is positive definite was reduced by Wall and Wetzel [18,19] to the question of whether a certain related sequence is a chain sequence.

Theorem 4.2 [19]. *The matrix A_n defined in (4.1) is positive definite if and only if $a_i > 0$, for $i = 1, \ldots, n$, and if*

$$\left\{\frac{b_1^2}{a_1a_2}, \frac{b_2^2}{a_2a_3}, \ldots, \frac{b_{n-1}^2}{a_{n-1}a_n}\right\}$$

is a chain sequence.

Hence, if we assume $A_n > 0$ in (4.1), then using Wall–Wetzel Theorem and Corollary 3.2, the eigenvalues of matrix J_n defined in (4.7) lie in the interval $[-1, 1]$, since

$$\alpha_n(-1) = \alpha_n(1) = \frac{b_{n-1}^2}{a_{n-1}a_n}.$$

From (4.6), the zeros of $Q_n(x)$ are in $\mathbb{R} \setminus [-1, 1]$. Taking into account the transformation (4.3) we may state the following theorem.

Theorem 4.3. *Given a tridiagonal matrix $A > 0$, all real zeros of $P_\mu(A)$ are located in a interval (a, b), with $-\infty < a < b < -1$.*

Due to the monotonicity of $Q_n(x)$ in the interval $[-1, 1]$ and from (4.3), it follows that Conjecture 1 is true for positive definite tridiagonal matrices.

We end this section with the following tridiagonal matrix

$$A = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1/2 & & & \\ & 1/2 & 1 & 1/2 & & \\ & & 1/2 & 1 & 1/2 & \\ & & & 1/2 & 2 & 1 \\ & & & & 1 & 1 \end{pmatrix}.$$

This matrix is positive definite and

$$P_\mu(A) = \frac{1}{4}\mu^3 + \frac{41}{16}\mu^2 + 6\mu + 4.$$

The eigenvalues of the Jacobi matrix

$$J = \begin{pmatrix} 0 & 1/\sqrt{2} & & & & \\ 1/\sqrt{2} & 0 & 1/2\sqrt{2} & & & \\ & 1/2\sqrt{2} & 0 & 1/2 & & \\ & & 1/2 & 0 & 1/2\sqrt{2} & \\ & & & 1/2\sqrt{2} & 0 & 1/\sqrt{2} \\ & & & & 1/\sqrt{2} & 0 \end{pmatrix}$$

are

$$\pm 0.886877, \ \pm 0.75851, \ \pm 0.371634,$$

with reciprocals, respectively,

$$\pm 1.127553, \ \pm 1.318373, \ \pm 2.690821,$$

which are the zeros of the polynomial defined in (4.4), whose negative squares are the zeros of $P_\mu(A)$:

$$-1.271375, \ -1.738108, \ -7.240517.$$

5. A particular case

In this final section we deal with the particular case where (4.1) is a Toeplitz matrix, i.e.,

$$B_n = \begin{pmatrix} a & b & & & \\ b & a & b & & \\ & b & \ddots & \ddots & \\ & & \ddots & \ddots & b \\ & & & b & a \end{pmatrix}_{n\times n}.$$

From (4.2), we have

$$P_\mu(B_n) = aP_\mu(B_{n-1}) + b^2 P_\mu(B_{n-2})\mu.$$

The eigenvalues of

$$J_n = \begin{pmatrix} 0 & \frac{b}{a} & & \\ \frac{b}{a} & 0 & \ddots & \\ & \ddots & \ddots & \frac{b}{a} \\ & & \frac{b}{a} & 0 \end{pmatrix}$$

are zeros of its characteristic polynomial

$$\frac{b^n}{a^n} U_n \left(\frac{ax}{2b} \right),$$

which are

$$\frac{2b}{a} \cos \left(\frac{k\pi}{n+1} \right),$$

for $k = 1, 2 \ldots, n$. Therefore, the zeros of the polynomial $P_\mu(B_n)$ are

$$-\frac{a^2}{4b^2} \left(\cos \left(\frac{k\pi}{n+1} \right) \right)^{-2},$$

for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

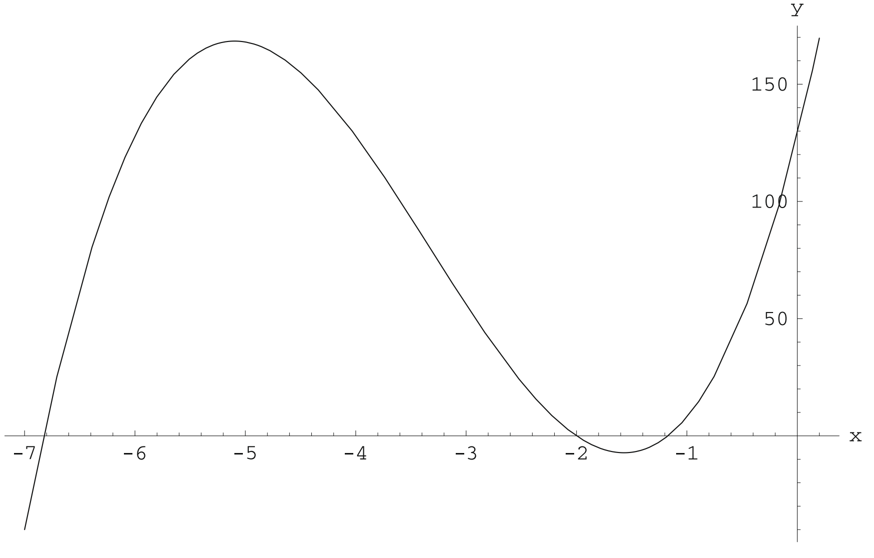
Example 5.1. For $n = 7$, $a = 2b = 2$, the zeros of the μ-polynomial

$$P_\mu(B_n) = 8\mu^3 + 80\mu^2 + 192\mu + 128$$

are

$$-2(2 + \sqrt{2}),\ -2,\ -2(2 - \sqrt{2}),$$

and the graph of is $P_\mu(B_n)$ is



References

[1] R.B. Bapat, Interpolating the determinantal and permanental Hadarmard inequality, Linear and Multilinear Algebra 32 (1992) 335–337.
[2] R.B. Bapat, A.K. Lal, Inequalities for the q-permanent, Linear Algebra Appl. 197/198 (1994) 397–409.
[3] M. Bożejko, R. Speicher, An example of a generalized Brownian motion, Comm. Math. Phys. 137 (1991) 519–531.
[4] E. Briand, On a matrix function interpolating between determinant and permanent, Linear Algebra Appl. 411 (2005) 343–347.
[5] T.S. Chihara, Chain sequences and orthogonal polynomials, Trans. Amer. Math. Soc. 10 (1962) 1–16.
[6] J. Favard, Sur les polynômes de Tchebicheff, C. R. Acad. Sci. Paris 200 (1935) 2052–2053.
[7] C.M. da Fonseca, On a conjecture about the μ-permanent, Linear and Multilinear Algebra 53 (2005) 225–230.
[8] C.M. da Fonseca, On the location of the eigenvalues of Jacobi matrices, Appl. Math. Lett. 19 (2006) 1168–1174.
[9] Mourad E.H. Ismail, D.R. Masson, J. Letessier, G. Valent, Birth and death processes and orthogonal polynomials, Orthogonal Polynomials (Columbus, OH, 1989), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 294, Kluwer Acad. Publ., Dordrecht, 1990, pp. 229–255.

[10] A.K. Lal, Coxeter Groups and Positive Matrices, Indian Statistical Institute, Delhi Center, India, 1992.
[11] A.K. Lal, Inequalities for the q-permanent. II, Linear Algebra Appl. 274 (1998) 1–16.
[12] M. Marcus, The permanent analogue of the Hadamard determinant theorem, Bull. Amer. Math. Soc. 69 (1963) 494–496.
[13] M. Marcus, H. Minc, Generalized matrix functions, Trans. Amer. Math. Soc. 116 (1965) 316–329.
[14] R. Merris, Immanantal invariants of graphs, Linear Algebra Appl. 401 (2005) 67–75.
[15] T. Muir, A Treatise on the Theory of Determinants, Dover, New York, 1960.
[16] G. Szegö, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, 11th ed., vol. 23, American Mathematical Society, Providence, RI, 2003.
[17] D. Vere-Jones, A generalization of permanents and determinants, Linear Algebra Appl. 111 (1988) 119–124.
[18] H.S. Wall, Analytic Theory of Continued Fractions, D. Van Nostrand Company, Inc., New York, NY, 1948.
[19] H.S. Wall, M. Wetzel, Quadratic forms and convergence regions for continued fractions, Duke Math. J. 11 (1944) 373–397.